Polynomials and Polynomial Functions

Algebra 2
Chapter 5

- This Slideshow was developed to accompany the textbook
  - *Larson Algebra 2*
  - *By Larson, R., Boswell, L., Kanold, T. D., & Stiff, L.*
  - *2011 Holt McDougal*
- Some examples and diagrams are taken from the textbook.

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5.1 Use Properties of Exponents

- When numbers get very big or very small, such as the mass of the sun = 5.98×10^{30} kg or the size of a cell = 1.0×10^{-6} m, we use scientific notation to write the numbers in less space than they normally would take.
- The properties of exponents will help you understand how to work with scientific notation.

5.1 Use Properties of Exponents

- What is an exponent and what does it mean?
  - A superscript on a number.
  - It tells the number of times the number is multiplied by itself.
- Example;
  - $x^3 = x \times x \times x$

Base Exponent

5.1 Use Properties of Exponents

- Properties of exponents
- $x^m \cdot x^n = x^{m+n}$ → product property
  - $x^2 \cdot x^3 =$
- $(xy)^m = x^m y^m$ → power of a product property
  - $(2 \cdot x)^3 =$
- $(x^m)^n = x^{mn}$ → power of a power property
  - $(2^3)^4 =$
- $\frac{x^m}{x^n} = x^{m-n}$ → quotient property
  - $\frac{x^4}{x^2} =$
- $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$ → power of a quotient property
  - $\left(\frac{4}{2}\right)^3 =$

5.1 Use Properties of Exponents

- $x^0 = 1$ → zero exponent property
- $x^{-m} = \frac{1}{x^m}$ → negative exponent property
  - $2^3 =$
  - $2^2 =$
  - $2^1 =$
  - $2^0 =$
  - $2^{-1} =$
  - $2^{-2} =$
  - $2^{-3} =$

5.1 Use Properties of Exponents

- $5^{-4}\,5^{3} =$
- $((-3)^2)^3 =$
- $(3^2x^2y)^2 =$

5.1 Use Properties of Exponents

- $\frac{12x^5a^2}{2x^4} \cdot \frac{2a}{3a^2}$
- $\frac{5x^2y^{-3}}{8x^{-4}} \cdot \frac{4x^{-3}y^2}{10x^{-2}z^0} =$

5.1 Use Properties of Exponents

- To multiply or divide scientific notation
  - think of the leading numbers as the coefficients and the power of 10 as the base and exponent.
- Example:
  - $2 \times 10^2 \cdot 5 \times 10^3 =$

5.2 Evaluate and Graph Polynomial Functions

- Large branches of mathematics spend all their time dealing with polynomials.
- They can be used to model many complicated systems.

5.2 Evaluate and Graph Polynomial Functions

- Polynomial in one variable
  - Function that has one variable and there are powers of that variable and all the powers are positive
- $4x^3 + 2x^2 + 2x + 5$
- $100x^{1234} - 25x^{345} + 2x + 1$
- $2/x$
- $3xy^2$

Not Polynomials in one variable.

5.2 Evaluate and Graph Polynomial Functions

- Degree
  - Highest power of the variable
- What is the degree?
  - $4x^3 + 2x^2 + 2x + 5$

5.2 Evaluate and Graph Polynomial Functions

- Types of Polynomial Functions
- Degree → Type
  - 0 → Constant → $y = 2$
  - 1 → Linear → $y = 2x + 1$
  - 2 → Quadratic → $y = 2x^2 + x - 1$
  - 3 → Cubic → $y = 2x^3 + x^2 + x - 1$
  - 4 → Quartic → $y = 2x^4 + 2x^2 - 1$

5.2 Evaluate and Graph Polynomial Functions

- Functions
  - $f(x) = 4x^3 + 2x^2 + 2x + 5$ means that this polynomial has the name f and the variable x
  - f(x) does not mean f times x!
- Direct Substitution
  - Example: find f(3)

5.2 Evaluate and Graph Polynomial Functions

- Synthetic Substitution
  - Example: find f(2) if $f(y) = -y^6 + 4y^4 + 3y^2 + 2y$

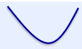
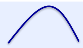
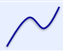
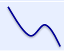
Coefficients with placeholders

2	-1	0	4	0	3	2	0
		-2	-4	0	0	6	16
	-1	-2	0	0	3	8	16

- f(2) = 16

5.2 Evaluate and Graph Polynomial Functions

- End Behavior
  - Polynomial functions always go towards ∞ or -∞ at either end of the graph

	Leading Coefficient +	Leading Coefficient -
Even Degree		
Odd Degree		

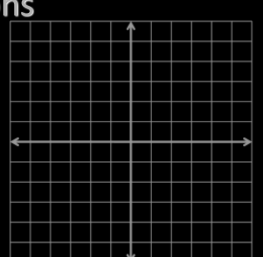
- Write
  - f(x) → +∞ as x → -∞ and f(x) → +∞ as x → +∞

5.2 Evaluate and Graph Polynomial Functions

- Graphing polynomial functions
  - Make a table of values
  - Plot the points
  - Make sure the graph matches the appropriate end behavior

5.2 Evaluate and Graph Polynomial Functions

Graph $f(x) = x^3 + 2x - 4$



5.3 Add, Subtract, and Multiply Polynomials

- Adding, subtracting, and multiplying are always good things to know how to do.
- Sometimes you might want to combine two or more models into one big model.

5.3 Add, Subtract, and Multiply Polynomials

- Adding and subtracting polynomials
  - Add or subtract the coefficients of the terms with the same power.
  - Called combining like terms.
- Examples:
  - $(5x^2 + x - 7) + (-3x^2 - 6x - 1)$
  - $(3x^3 + 8x^2 - x - 5) - (5x^3 - x^2 + 17)$

5.3 Add, Subtract, and Multiply Polynomials

- Multiplying polynomials
  - Use the distributive property
- Examples:
  - $(x - 3)(x + 4)$
- $(x + 2)(x^2 + 3x - 4)$

5.3 Add, Subtract, and Multiply Polynomials

- $(x - 1)(x + 2)(x + 3)$

5.3 Add, Subtract, and Multiply Polynomials

- Special Product Patterns
  - Sum and Difference
    - $(a - b)(a + b) = a^2 - b^2$
  - Square of a Binomial
    - $(a \pm b)^2 = a^2 \pm 2ab + b^2$
  - Cube of a Binomial
    - $(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$

5.3 Add, Subtract, and Multiply Polynomials

- $(x + 2)^3$
- $(x - 3)^2$

5.4 Factor and Solve Polynomial Equations

- A manufacturer of shipping cartons who needs to make cartons for a specific use often has to use special relationships between the length, width, height, and volume to find the exact dimensions of the carton.
- The dimensions can usually be found by writing and solving a polynomial equation.
- This lesson looks at how factoring can be used to solve such equations.

5.4 Factor and Solve Polynomial Equations

How to Factor

1. Greatest Common Factor
   - Comes from the distributive property
   - If the same number or variable is in each of the terms, you can bring the number to the front times everything that is left.
   - $3x^2y + 6xy - 9xy^2 =$
   - Look for this first!

5.4 Factor and Solve Polynomial Equations

2. Check to see how many terms
   - Two terms
     - Difference of two squares: $a^2 - b^2 = (a - b)(a + b)$
       - $9x^2 - y^4 =$
     - Sum of Two Cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
       - $8x^3 + 27 =$
     - Difference of Two Cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
       - $y^3 - 8 =$

5.4 Factor and Solve Polynomial Equations

- Three terms
  - General Trinomials → $ax^2 + bx + c$

1. Write two sets of parentheses ( )( )
2. Guess and Check
3. The Firsts multiply to make ax^2
4. The Lasts multiply to make c
5. The Outers + Inners make bx
   - $x^2 + 7x + 10 =$
   - $x^2 + 3x - 18 =$
   - $6x^2 - 7x - 20 =$

5.4 Factor and Solve Polynomial Equations

- Four terms
  - Grouping
    - Group the terms into sets of two so that you can factor a common factor out of each set
    - Then factor the factored sets (Factor twice)
    - $b^3 - 3b^2 - 4b + 12 =$

5.4 Factor and Solve Polynomial Equations

3. Try factoring more!

- Examples:
  - $a^2x - b^2x + a^2y - b^2y =$

5.4 Factor and Solve Polynomial Equations

- $3a^2z - 27z =$
- $n^4 - 81 =$

5.4 Factor and Solve Polynomial Equations

- Solving Equations by Factoring
  - Make = 0
  - Factor
  - Make each factor = 0 because if one factor is zero, 0 time anything = 0

5.4 Factor and Solve Polynomial Equations

- $2x^5 = 18x$

5.5 Apply the Remainder and Factor Theorems

- So far we done add, subtracting, and multiplying polynomials.
- Factoring is similar to division, but it isn't really division.
- Today we will deal with real polynomial division.

5.5 Apply the Remainder and Factor Theorems

- Long Division
  - Done just like long division with numbers
- $\frac{y^4+2y^2-y+5}{y^2-y+1}$

5.5 Apply the Remainder and Factor Theorems

- $\frac{x^3+4x^2-3x+10}{x+2}$

5.5 Apply the Remainder and Factor Theorems

- Synthetic Division
  - Shortened form of long division for dividing by a **binomial**
  - Only when dividing by (x – r)

5.5 Apply the Remainder and Factor Theorems

- Synthetic Division
  - Example: $(-5x^5 - 21x^4 - 3x^3 + 4x^2 + 2x + 2) / (x + 4)$

Coefficients with placeholders

-4	-5	-21	-3	4	2	2
		20	4	-4	0	-8
	-5	-1	1	0	2	-6

$$-5x^4 - x^3 + x^2 + 2 + \frac{-6}{x+4}$$

5.5 Apply the Remainder and Factor Theorems

- $(2y^5 + 64)(2y + 4)^{-1}$

$$\frac{2y^5+64}{2y+4} \rightarrow \frac{y^5+32}{y+2}$$

-2	1	0	0	0	0	32
		-2	4	-8	16	-32
	1	-2	4	-8	16	0

- $y^4 - 2y^3 + 4y^2 - 8y + 16$

5.5 Apply the Remainder and Factor Theorems

- Remainder Theorem
  - if polynomial $f(x)$ is divided by the binomial $(x - a)$, then the remainder equals $f(a)$.
  - Synthetic substitution
  - Example: if $f(x) = 3x^4 + 6x^3 + 2x^2 + 5x + 9$, find $f(9)$
    - Use synthetic division using $(x - 9)$ and see remainder.

5.5 Apply the Remainder and Factor Theorems

- The Factor Theorem
  - The binomial x – a is a factor of the polynomial f(x) iff f(a) = 0

5.5 Apply the Remainder and Factor Theorems

- Using the factor theorem, you can find the factors (and zeros) of polynomials
- Simply use synthetic division using your first zero (you get these off of problem or off of the graph where they cross the x-axis)
- The polynomial answer is one degree less and is called the depressed polynomial.
- Divide the depressed polynomial by the next zero and get the next depressed polynomial.
- Continue doing this until you get to a quadratic which you can factor or use the quadratic formula to solve.

5.5 Apply the Remainder and Factor Theorems

- Show that x – 2 is a factor of $x^3 + 7x^2 + 2x - 40$. Then find the remaining factors.

5.6 Find Rational Zeros

- Rational Zero Theorem
  - Given a polynomial function, the rational zeros will be in the form of p/q where p is a factor of the last (or constant) term and q is the factor of the leading coefficient.

5.6 Find Rational Zeros

- List all the possible rational zeros of
- $f(x) = 2x^3 + 2x^2 - 3x + 9$

5.6 Find Rational Zeros

- Find all rational zeros of $f(x) = x^3 - 4x^2 - 2x + 20$

5.7 Apply the Fundamental Theorem of Algebra

- When you are finding the zeros, how do you know when you are finished?
- Today we will learn about how many zeros there are for each polynomial function.

5.7 Apply the Fundamental Theorem of Algebra

- Fundamental Theorem of Algebra
  - A polynomial function of degree greater than zero has at least one zero.
  - These zeros may be imaginary however.
  - There is the same number of zeros as there is degree – you may have the same zero more than once though.
    - Example $x^2 + 6x + 9=0 \rightarrow (x + 3)(x + 3)=0 \rightarrow$ zeros are -3 and -3

5.7 Apply the Fundamental Theorem of Algebra

- Complex Conjugate Theorem
  - If the complex number a + bi is a zero, then a – bi is also a zero.
  - Complex zeros come in pairs
- Irrational Conjugate Theorem
  - If $a+\sqrt{b}$ is a zero, then so is $a-\sqrt{b}$

5.7 Apply the Fundamental Theorem of Algebra

- Given a function, find the zeros of the function.

$$f(x) = x^3 - 7x^2 + 16x - 10$$

5.7 Apply the Fundamental Theorem of Algebra

- Write a polynomial function that has the given zeros. 2, 4i

5.7 Apply the Fundamental Theorem of Algebra

- Descartes' Rule of Signs
  - If f(x) is a polynomial function, then
    - The number of **positive** real zeros is equal to the number of sign changes in **f(x)** or less by even number.
    - The number of **negative** real zeros is equal to the number of sign changes in **f(-x)** or less by even number.

5.7 Apply the Fundamental Theorem of Algebra

- Determine the possible number of positive real zeros, negative real zeros, and imaginary zeros for $g(x) = 2x^4 - 3x^3 + 9x^2 - 12x + 4$
  - Positive zeros:
    - 4, 2, or 0
  - Negative zeros: $g(-x) = 2x^4 + 3x^3 + 9x^2 + 12x + 4$
    - 0

Positive	Negative	Imaginary	Total
4	0	0	4
2	0	2	4
0	0	4	4

5.8 Analyze Graphs of Polynomial Functions

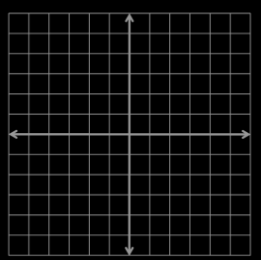
- If we have a polynomial function, then
  - k is a zero or root
  - k is a solution of f(x) = 0
  - k is an x-intercept if k is real
  - x – k is a factor

5.8 Analyze Graphs of Polynomial Functions

- Use x-intercepts to graph a polynomial function
- $f(x) = ½ (x + 2)^2(x – 3)$
  - since (x + 2) and (x – 3) are factors of the polynomial, the x-intercepts are -2 and 3
  - plot the x-intercepts
  - Create a table of values to finish plotting points around the x-intercepts
  - Draw a smooth curve through the points

5.8 Analyze Graphs of Polynomial Functions

- Graph $f(x) = ½ (x + 2)^2(x – 3)$

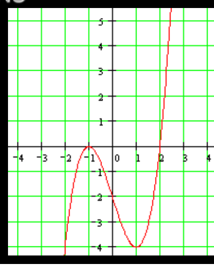


5.8 Analyze Graphs of Polynomial Functions

- Turning Points
  - Local Maximum and minimum (turn from going up to down or down to up)
  - The graph of every polynomial function of degree n can have at most n-1 turning points.
  - If a polynomial function has n distinct real zeros, the function will have exactly n-1 turning points.
  - Calculus lets you find the turning points easily.

5.8 Analyze Graphs of Polynomial Functions

- What are the turning points?



5.9 Write Polynomial Functions and Models

- You keep asking, “Where will I ever use this?” Well today we are going to model a few situations with polynomial functions.

5.9 Write Polynomial Functions and Models

- Writing a function from the x-intercepts and one point
  - Write the function as factors with an *a* in front
  - y = a(x – p)(x – q)…
  - Use the other point to find a
- Example:
  - x-intercepts are -2, 1, 3 and (0, 2)

5.9 Write Polynomial Functions and Models

- Show that the nth-order differences for the given function of degree n are nonzero and constant.
  - Find the values of the function for equally spaced intervals
  - Find the differences of the *y* values
  - Find the differences of the differences and repeat until all are the same value

5.9 Write Polynomial Functions and Models

- Show that the 3rd order differences are constant of $f(x) = 2x^3 + x^2 + 2x + 1$

5.9 Write Polynomial Functions and Models

- Finding a model given several points
  - Find the degree of the function by finding the finite differences
    - Degree = order of constant nonzero finite differences
  - Write the basic standard form functions (i.e. $f(x) = ax^3 + bx^2 + cx + d$)
  - Fill in x and f(x) with the points
  - Use some method to find a, b, c, and d
    - Cramer's rule or graphing calculator using matrices or computer program

5.9 Write Polynomial Functions and Models

- Find a polynomial function to fit:
- f(1) = -2, f(2) = 2, f(3) = 12, f(4) = 28, f(5) = 50, f(6) = 78

5.9 Write Polynomial Functions and Models

- Regressions on TI Graphing Calculator

1. Push STAT ↓ Edit…
2. Clear lists, then enter x's in 1st column and y's in 2nd
3. Push STAT → CALC ↓ (regression of your choice)
4. Push ENTER twice
5. Read your answer

5.9 Write Polynomial Functions and Models

- Regressions using Microsoft Excel

1. Enter x's and y's into 2 columns
2. Insert X Y Scatter Chart
3. In Chart Tools: Layout pick Trendline → More Trendline options
4. Pick a Polynomial trendline and enter the degree of your function AND pick Display Equation on Chart
5. Click Done
6. Read your answer off of the chart.